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Widths of Besov classes of generalized smoothness on the sphere

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ABSTRACT

We introduce the Besov spaces $B^{\Omega}_{p\theta}(\mathbb{S}^{d-1})$ of generalized smoothness on the sphere $\mathbb{S}^{d-1}$, and obtain the representation theorem, an embedding theorem, and the characterization using a frame. We also study the Kolmogorov, linear and Gelfand widths of Besov classes $BB^{\Omega}_{p\theta}(\mathbb{S}^{d-1})$ of generalized smoothness in $L_q(\mathbb{S}^{d-1})$ for $1 \le p, q \le \infty$, and obtain their asymptotic orders.

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1. Introduction

This paper is devoted to discussing the representation and approximation on the sphere of function spaces of generalized smoothness. Besov spaces of generalized smoothness have been investigated by many authors in different contexts, and there has been an increasing interest in recent years. On the one hand, this interest is in connection with embeddings, limiting embeddings and approximations; on the other hand, it is also connected with applications in probability theory and the theory of stochastic processes and the study of trace spaces on fractals, that is, so-called h-sets. We refer the reader here to Farkas and Leopold [10], Cobos and Kühn [6], Sun and Wang [27], Haroske and Moura [11], Moura et al. [22], Moura [21], and Bricchi [3]. However, all spaces in these papers are defined on $\mathbb{R}^d$, or on a non-empty bounded domain of $\mathbb{R}^d$ with sufficiently smooth boundary, or on the d-dimensional torus, but not on the sphere.

The aim of this paper is twofold. The first is to give the definition and characterization of Besov spaces on the sphere of generalized smoothness, and the second one is to obtain the asymptotic

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orders of the Kolmogorov, linear and Gelfand widths of Besov classes of generalized smoothness on the sphere for all possible cases.

We organize this paper as follows. Section 2 contains notation, definitions, and main results about widths. In Section 3, we prove representation theorems for the Besov spaces of generalized smoothness and by means of this obtain an embedding theorem and the best approximation by spherical polynomials. The characterization of the generalized Besov spaces using a frame is given in Section 4 and discretization of the problem of estimates of widths is introduced in Section 5. Finally, we prove the main results concerning widths in Section 6.

2. Preliminaries and main results

Let $\mathbb{S}^{d-1} := \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d : x_1^2 + \cdots + x_d^2 = 1\}$ be the unit sphere of $\mathbb{R}^d$ equipped with the Lebesgue measure $d\sigma$. We denote by $\|f\|_p$ the quantity $\left(\int_{\mathbb{S}^{d-1}} |f(x)|^p d\sigma(x)\right)^{\frac{1}{p}}$ for $1 \le p < \infty$ and by $\|f\|_\infty$ the essential supremum of $|f|$ over $\mathbb{S}^{d-1}$. Thus, for $1 \le p < \infty, L_p \equiv L_p(\mathbb{S}^{d-1}) = \{f : \|f\|_p < \infty\}$ is a linear space equipped with the norm $\|\cdot\|_p$, and for $p = \infty$ we assume that L_∞ is the space $C(\mathbb{S}^{d-1})$ of continuous functions on $\mathbb{S}^{d-1}$ with the norm $\|\cdot\|_\infty$. Denote by Π_N^d the space of all spherical polynomials of degree at most N on $\mathbb{S}^{d-1}$ (i.e., polynomials in d variables of total degree at most N restricted to $\mathbb{S}^{d-1}$), by $\mathcal{H}_l^d$ the space of all spherical harmonic polynomials of degree l on $\mathbb{S}^{d-1}$, and by H_l^d $(l = 0, 1, \ldots)$ the orthogonal projection operator from $L_2(\mathbb{S}^{d-1})$ onto $\mathcal{H}_l^d$. It is well known that the spaces $\mathcal{H}_l^d$, $l = 0, 1, 2, \ldots$, are just the eigenspaces corresponding to the eigenvalues $-l(l + d - 1)$ of the Laplace–Beltrami operator $\triangle$. Given $r \ge 0$, we define the rth-order Laplace–Beltrami operator $(-\triangle)^r$ on $\mathbb{S}^{d-1}$ in a distributional sense by

$$H_l^d((-\triangle)^r(f)) = (l(l + d - 1))^r H_l^d(f), \quad l = 0, 1, 2, \ldots,$$

where f is a distribution on $\mathbb{S}^{d-1}$. For $f \in L_p$, if there exists a function $g \in L_p$ such that $H_l^d(g) = (l(l + d - 1))^{r/2} H_l^d(f)$ for all $l \in \mathbb{N}$, then we call g the rth-order derivative of f in the sense of L_p and write $g = f^{(r)} = (-\triangle)^{r/2} f$. Clearly, $\Pi_N^d = \sum_{k=0}^N \mathcal{H}_k^d$ and $\dim \Pi_N^d = \sum_{k=0}^N \dim \mathcal{H}_k^d \asymp N^{d-1}$. For $f \in L_p$, we define

$$E_N(f)_p := \inf\left\{\|f - T\|_p : T \in \Pi_N^d\right\}, \quad N \in \mathbb{N}.$$

In this paper, we want to discuss characterization and approximation of functions of generalized smoothness on the sphere. First, we introduce some definitions and notation.

Let I denote the identity operator, and S_h denote the spherical translation operator, i.e.,

$$S_h(f)(\xi) = \frac{1}{\ell(h)} \int_{\ell_{\xi,h}} f(\eta) d\ell_{\xi,h}(\eta),$$

where $\ell_{\xi,h} = \{x \in \mathbb{S}^{d-1} : x \cdot \xi = \cos h\}$, $d\ell_{\xi,h}$ is the surface measure of $\ell_{\xi,h}$, and $\ell(h)$ is the surface area of $\ell_{\xi,h}$. For all $r \ge 0$, we set

$$\triangle_h^r = (I - S_h)^{\frac{r}{2}} = \sum_{i=0}^\infty (-1)^i \binom{\frac{r}{2}}{i} (S_h)^i, \quad \binom{\frac{r}{2}}{i} = \frac{1}{i!} \frac{r}{2} \left(\frac{r}{2} - 1\right) \cdots \left(\frac{r}{2} - i + 1\right).$$

When r is not an even integer, $\triangle_h^r$ is the fractional power of the difference operator $I - S_h$. For $f \in L_p$, the modulus of smoothness of degree r of f is defined by

$$\omega_r(f, t)_p = \sup\left\{\| \triangle_h^r (f)\|_p : 0 < h \le t\right\}, \quad 0 < t \le \pi.$$

Some of the properties of $\omega_r(f, t)_p$ are collected below (see [26], [32, p. 184]):

(1) $\lim_{t\to 0} \omega_r(f, t)_p = 0$;
(2) $\omega_r(f, t)_p \le \omega_r(f, s)_p$, $0 < t < s \le \pi$;
(3) $\omega_r(f + g, t)_p \le \omega_r(f, t)_p + \omega_r(g, t)_p$;

(4) $\omega_r(f, t)_p \le 2^{\left[\frac{r-m+1}{2}\right]}\omega_m(f, t),\ 0 < m < r$;
(5) $\omega_r(f, Nt)_p \le cN^r\omega_r(f, t)_p$, where N is a positive integer, and c is a constant independent of $f,\ t$ and N.

The following Jackson type inequalities relating to the best polynomial approximation on the sphere are fundamental: for $f \in W_p^\beta(\mathbb{S}^{d-1}),\ \beta \ge 0$,

$$E_N(f)_p \le CN^{-\beta}\omega_r(f^{(\beta)}, 1/N)_p, \tag{2.1}$$

where C is a positive constant independent of f and N. For $\beta = 0$, (2.1) was established with great effort by many people (see [26], [32, p. 194, Theorem 5.1.1] and related citations), while for $\beta > 0$, the inequality (2.1) is a direct consequence of the equivalence of the modulus of smoothness and the K-functional, (2.1) with $\beta = 0$, and Theorem 7.2 in [9].

Next we introduce Besov spaces $B_{p,\theta}^\Omega(\mathbb{S}^{d-1})$ of generalized smoothness on the sphere. Let $l \ge 1$ be a fixed positive number and let Ω denote a nonnegative function on $\mathbb{R}_+ = \{t : t \ge 0\}$. We say that $\Omega(t) \in \Phi_l^*$ if it satisfies:

(1) $\Omega(0) = 0$ and $\Omega(t) > 0$ for any $t > 0$;
(2) $\Omega(t)$ is continuous on $\mathbb{R}_+$;
(3) $\Omega(t)$ is almost increasing on $\mathbb{R}_+$, i.e., for any t, τ with $0 \le t \le \tau$, we have $\Omega(t) \le C\Omega(\tau)$, where $C \ge 1$ is a constant independent of t and τ;
(4) for any $n \in \mathbb{Z}_+$ and $t > 0$, $\Omega(nt) \le Cn^l\Omega(t)$, where $C > 0$ is a constant independent of n and t;
(5) there exists $\alpha > 0$ such that $\Omega(t)/t^\alpha$ is almost increasing on $\mathbb{R}_+$;
(6) there exists $0 < \beta < l$ such that $\Omega(t)/t^\beta$ is almost decreasing on $\mathbb{R}_+$, i.e., there exists $C > 0$ such that for any $t,\ \tau$ with $0 < t \le \tau$, we have

$$\Omega(t)/t^\beta \ge C\Omega(\tau)/\tau^\beta.$$

A prototype of functions of Φ_l^* is $\Omega(t) = t^\alpha(1 + (\ln 1/t)_+)^\beta,\ 0 < \alpha < l,\ \beta \in \mathbb{R}$. We note that the ideas of almost increasing and almost decreasing are due to Bary and Stechkin [1].

Now let $\Omega(t) \in \Phi_l^*$. We say that $f \in B_{p,\theta}^\Omega \equiv B_{p,\theta}^\Omega(\mathbb{S}^{d-1}),\ 1 \le p, \theta \le \infty$ if f satisfies the following conditions:

(1) $f \in L_p(\mathbb{S}^{d-1})$;
(2) $$\|f\|_{b_{p,\theta}^\Omega} = \begin{cases} \left\{\int_0^{+\infty}\left(\frac{\omega_l(f, t)_p}{\Omega(t)}\right)^\theta \frac{dt}{t}\right\}^{\frac{1}{\theta}} < \infty, & 1 \le \theta < \infty; \\ \sup_{t>0}\frac{\omega_l(f, t)_p}{\Omega(t)} < \infty, & \theta = \infty. \end{cases}$$

Then the space $B_{p,\theta}^\Omega(\mathbb{S}^{d-1})$ is a Banach space with the norm

$$\|f\|_{B_{p,\theta}^\Omega} := \|f\|_p + \|f\|_{b_{p,\theta}^\Omega}.$$

Denote by $BB_{p,\theta}^\Omega \equiv BB_{p,\theta}^\Omega(\mathbb{S}^{d-1})$ the unit ball of $B_{p,\theta}^\Omega(\mathbb{S}^{d-1})$. Note that when $\Omega(t) = t^\alpha,\ \alpha > 0$, the space $B_{p,\theta}^\Omega(\mathbb{S}^{d-1})$ coincides with the usual Besov space $B_{p,\theta}^\alpha(\mathbb{S}^{d-1})$, which was first introduced in [16], and therein a series of equivalent norms of $B_{p,\theta}^\alpha(\mathbb{S}^{d-1})$ were given. We shall give representation theorems, an embedding theorem, and the characterization using a frame of the space $B_{p,\theta}^\Omega(\mathbb{S}^{d-1})$ in Sections 3 and 4.

For a given subset K of a normed linear space $(X, \|\cdot\|)$, the Kolmogorov n-width $d_n(K, X)$ is defined by

$$d_n(K, X) = \inf_{L_n}\sup_{x\in K}\inf_{y\in L_n}\|x - y\|,$$

with the leftmost infimum being taken over all n-dimensional linear subspaces L_n of X. The linear n-width is defined by

$$\delta_n(K, X) = \inf_{P_n}\sup_{f\in K}\|f - P_nf\|,$$

with the infimum being taken over all linear continuous operators P_n on X with rank $\le n$. We say that a subspace $X^M \subset X$ is of codimension M if there exist M linearly independent continuous linear functionals $\lambda_1, \ldots, \lambda_M$ on X such that

$$X^M = \{x \in X \,:\, \lambda_i(x) = 0,\ i = 1, \ldots, M\}, \qquad X^0 := X.$$

The Gelfand n-width $d^n(K, X)$ is defined by

$$d^n(K, X) = \inf_{X^n} \sup \left\{ \|x\| \,:\, x \in K \cap X^n \right\},$$

with the infimum being taken over all subspaces $X^n \subset X$ of codimension $\leq n$. More information on Kolmogorov, linear, and Gelfand widths can be found in [25,17].

One of our main purpose in this paper is to consider the asymptotic orders of the Kolmogorov n-widths $d_n\left(BB^{\Omega}_{p,\theta}(\mathbb{S}^{d-1}), L_q(\mathbb{S}^{d-1})\right)$, the linear n-widths $\delta_n\left(BB^{\Omega}_{p,\theta}(\mathbb{S}^{d-1}), L_q(\mathbb{S}^{d-1})\right)$, and the Gelfand widths $d^n\left(BB^{\Omega}_{p,\theta}(\mathbb{S}^{d-1}), L_q(\mathbb{S}^{d-1})\right)$ for all $1 \leq p, q \leq \infty$ as $n \to \infty$. In the periodic case, the exact orders of the Kolmogorov and linear widths of smooth function classes in L_q space can be found in [29,27]. The Kolmogorov, Gelfand, and linear widths of embeddings in function spaces of Besov and Triebel–Lizorkin type on the bounded Lipschitz domain were studied in [31]. We note that on the sphere the Kolmogorov n-widths and the linear n-widths of the Sobolev classes $BW^r_p(\mathbb{S}^{d-1})$ in $L_q(\mathbb{S}^{d-1})$ were studied in [14,15,4,2,5], in which optimal asymptotic orders of $d_n\left(BW^r_p(\mathbb{S}^{d-1}), L_q(\mathbb{S}^{d-1})\right)$ and $\delta_n\left(BW^r_p(\mathbb{S}^{d-1}), L_q(\mathbb{S}^{d-1})\right)$ as $n \to \infty$ are given for all $1 \leq p, q \leq \infty$. In [13,18], the complexity of quadrature and the ε-entropy in the usual Besov spaces on the sphere were established.

Our main results concerning widths then can be formulated as follows:

Theorem 1. *Let $\Omega(t) = t^{\alpha}\Omega_1(t),\ \Omega(t),\ \Omega_1(t) \in \Phi^*_l,\ 1 \leq p, q, \theta \leq \infty$. Then*

$$d_n(BB^{\Omega}_{p,\theta}, L_q) \asymp \Omega\left(n^{-\frac{1}{d-1}}\right) n^{\lambda_1}, \quad \text{if } \alpha > \tau_1,$$

$$d^n(BB^{\Omega}_{p,\theta}, L_q) \asymp \Omega\left(n^{-\frac{1}{d-1}}\right) n^{\lambda_2}, \quad \text{if } \alpha > \tau_2,$$

$$\delta_n(BB^{\Omega}_{p,\theta}, L_q) \asymp \Omega\left(n^{-\frac{1}{d-1}}\right) n^{\lambda_3}, \quad \text{if } \alpha > \tau_3,$$

where $A(n) \asymp B(n)$ means that $A(n) \ll B(n)$ and $A(n) \gg B(n)$, and $A(n) \ll B(n)$ means that there exists a positive constant c independent of n such that $A(n) \leq cB(n)$, and:

(1) $\lambda_i = \tau_i = 0,\ i = 1, 2, 3$, *if p, q lie in the region I: $1 \leq q \leq p \leq \infty$;*
(2) $\lambda_1 = \lambda_3 = 1/p - 1/q,\ \lambda_2 = 0,\ \tau_1 = \tau_3 = (1/p - 1/q)(d-1),\ \tau_2 = (d-1)/2$ *if p, q lie in the region II: $1 \leq p \leq q \leq 2$;*
(3) $\lambda_1 = 0,\ \lambda_2 = \lambda_3 = 1/p - 1/q,\ \tau_1 = (d-1)/2,\ \tau_2 = \tau_3 = (1/p - 1/q)(d-1)$ *if p, q lie in the region III: $2 \leq p \leq q \leq \infty$;*
(4) $\lambda_1 = 1/p - 1/2,\ \lambda_2 = 1/2 - 1/q,\ \lambda_3 = \max(1/p - 1/2, 1/2 - 1/q),\ \tau_1 = (d-1)/p,\ \tau_2 = (1 - 1/q)(d-1),\ \tau_3 = 2(d-1)\max(1/p,\ 1 - 1/q) - (d-1)$ *if p, q lie in the region IV: $1 \leq p \leq 2 \leq q \leq \infty$.*

3. The representation theorem and the embedding theorem for $B^{\Omega}_{p,\theta}(\mathbb{S}^{d-1})$

Let $\eta \in C^{\infty}[0, \infty)$ and $\chi_{[0,1]} \leq \eta \leq \chi_{[0,2)}$. We define the operators V_N, $N \in \mathbb{N}$, by

$$V_0(f) = H^d_0(f), \qquad V_N(f) = \sum_{k=0}^{\infty} \eta\left(\frac{k}{N}\right) H^d_k(f), \quad N \geq 1,\ f \in L_p(\mathbb{S}^{d-1}),\ 1 \leq p \leq \infty,$$

where $H^d_k(f)$ denotes the orthogonal projection of f onto $\mathcal{H}^d_k$. Then the following assertions are true (see, for example, [32, p. 161]):

(a) $V_N(f) \in \Pi^d_{2N-1}$;
(b) $V_N(f) = f$, if $f \in \Pi^d_N$;
(c) $\|V_N(f)\|_p \leq c\|f\|_p$;
(d) $\|f - V_N(f)\|_p \leq cE_N(f)_p$,

where $c > 0$ is a constant depending only on the function η and the dimension d.

For $f \in L_p$, we define

$$A_0(f) = V_0(f), \qquad A_1(f) = V_1(f), \qquad A_s(f) = V_{2^{s-1}}(f) - V_{2^{s-2}}(f) \quad \text{for } s \geq 2.$$

Then $A_s(f) \in \Pi^d_{2^s}$, and $\sum_{s=0}^{\infty} A_s(f)$ converges to f in L_p norm.

The following representation theorem is fundamental. Similar results on the torus can be seen in [27].

Theorem 2. *. Let $\Omega(t) \in \Phi^*_l,\ 1 \leq p, \theta \leq \infty,\ l > 0$. Then the following conditions are equivalent:*

(1°) $f \in B^{\Omega}_{p,\theta}$;

(2°) $$\|f\|^{(1)}_{B^{\Omega}_{p,\theta}} := \begin{cases} \left\{\sum_{s=0}^{\infty} \left(\frac{\|A_s(f)\|_p}{\Omega(2^{-s})}\right)^{\theta}\right\}^{\frac{1}{\theta}}, & 1 \leq \theta < \infty; \\ \sup_{s\geq 0} \frac{\|A_s(f)\|_p}{\Omega(2^{-s})}, & \theta = \infty \end{cases} < +\infty;$$

(3°) $$\|f\|^{(2)}_{B^{\Omega}_{p,\theta}} := \begin{cases} \|f\|_p + \left\{\sum_{s=0}^{\infty} \left(\frac{E_{2^s}(f)_p}{\Omega(2^{-s})}\right)^{\theta}\right\}^{\frac{1}{\theta}}, & 1 \leq \theta < \infty; \\ \|f\|_p + \sup_{s\geq 0} \frac{E_{2^s}(f)_p}{\Omega(2^{-s})}, & \theta = \infty \end{cases} < +\infty.$$

Proof. (2°) $\Rightarrow$ (1°). It suffices to prove

$$\|f\|_{B^{\Omega}_{p,\theta}} \ll \|f\|^{(1)}_{B^{\Omega}_{p,\theta}}. \tag{3.1}$$

Since $\Omega(t)$ is almost increasing and $\Omega(t)/t^{\beta}$ is almost decreasing on $(0, \infty)$ for some $\beta > 0$, we get for any $t_0 > 0$ and $t \in [t_0, 2t_0]$,

$$\Omega(t) \asymp \Omega(t_0). \tag{3.2}$$

For $1 \leq \theta < \infty$,

$$\|f\|^{\theta}_{b^{\Omega}_{p,\theta}} = \left(\int_0^1 + \int_1^{\infty}\right) \left(\frac{\omega_l(f,\ t)_p}{\Omega(t)}\right)^{\theta} \frac{dt}{t} =: I_1 + I_2. \tag{3.3}$$

Let us estimate I_1. We have by (3.2)

$$\begin{aligned} I_1 &= \int_0^1 \left(\frac{\omega_l(f,\ t)_p}{\Omega(t)}\right)^{\theta} \frac{dt}{t} = \ln 2 \int_0^{\infty} \left(\frac{\omega_l(f,\ 2^{-u})_p}{\Omega(2^{-u})}\right)^{\theta} du \\ &= \ln 2 \sum_{N=0}^{+\infty} \int_N^{N+1} \left(\frac{\omega_l(f,\ 2^{-u})_p}{\Omega(2^{-u})}\right)^{\theta} du \ll \sum_{N=0}^{+\infty} \left(\frac{\omega_l(f,\ 2^{-N})_p}{\Omega(2^{-N})}\right)^{\theta}. \end{aligned}$$

By the triangle inequality we get that

$$\| \triangle^l_t(f)\|_p = \left\| \triangle^l_t \left(\sum_{s=0}^{\infty} A_s(f)\right) \right\|_p \leq \sum_{s=0}^{\infty} \| \triangle^l_t(A_s(f))\|_p, \quad t > 0. \tag{3.4}$$

Using the fact that (see [32, Theorem 4.6.1, p. 188])

$$\| \triangle^l_t T_m\|_p \ll t^l \|T^{(l)}_m\|_p, \quad T_m \in \Pi^d_m,\ l > 0,$$

and the Bernstein inequality (see for example [7])

$$\|T^{(l)}_m\|_p \ll m^l \|T_m\|_p, \quad T_m \in \Pi^d_m,\ l > 0,$$

we have

$$\| \triangle^l_t(A_s(f))\|_p \ll t^l \|(A_s(f))^{(l)}\|_p \ll t^l 2^{sl} \|A_s(f)\|_p. \tag{3.5}$$

On the other hand, it follows from the definition of $\triangle_t^l$ that

$$\| \triangle_t^l(A_s(f))\|_p \le 2^l \|A_s(f)\|_p. \tag{3.6}$$

Clearly, (3.5) and (3.6) imply that

$$\sup_{t\le 2^{-N}} \| \triangle_t^l\ (A_s(f))\|_p \ll \min\{1, 2^{(s-N)l}\}\|A_s(f)\|_p,$$

which, combining with (3.4), results in

$$\omega_l(f, 2^{-N})_p \ll \sum_{s=0}^{+\infty} \min\{1, 2^{(s-N)l}\}\|A_s(f)\|_p.$$

Then,

$$I_1 \ll \sum_{N=0}^{+\infty}\left(\frac{\omega_l(f,\ 2^{-N})_p}{\Omega(2^{-N})}\right)^\theta \ll \sum_{N=0}^{+\infty}\left(\frac{1}{\Omega(2^{-N})}\right)^\theta \left(\sum_{s=0}^{N} 2^{(s-N)l}\|A_s(f)\|_p\right)^\theta + \sum_{N=0}^{+\infty}\left(\frac{1}{\Omega(2^{-N})}\right)^\theta \left(\sum_{s=N+1}^{+\infty} \|A_s(f)\|_p\right)^\theta =: J_1 + J_2. \tag{3.7}$$

Suppose that $\alpha,\ \beta\ \in\ (0, l)$ are numbers satisfying Conditions (5) and (6) in the definition of $\Omega(t) \in \Phi_l^*$. We choose $\delta > 0$ such that $\delta + \beta < l$. Then,

$$\begin{aligned} J_1 &= \sum_{N=0}^{+\infty}\left(\frac{1}{\Omega(2^{-N})}\right)^\theta \left(\sum_{s=0}^{N} 2^{\delta(s-N)}2^{(s-N)(l-\delta)}\|A_s(f)\|_p\right)^\theta \\ &\ll \sum_{N=0}^{+\infty}\left(\frac{1}{\Omega(2^{-N})}\right)^\theta \sum_{s=0}^{N} 2^{(s-N)(l-\delta)\theta}\|A_s(f)\|_p^\theta \\ &= \sum_{s=0}^{+\infty}\|A_s(f)\|_p^\theta \sum_{N=s}^{+\infty}\left(\frac{2^{(s-N)(l-\delta)}}{\Omega(2^{-N})}\right)^\theta, \end{aligned}$$

where in the first inequality, we used the Hölder inequality, and in the last equality we interchanged the order of the sums. By Condition (6), we know that $\Omega(t)/t^\beta$ is almost decreasing. Thus,

$$\frac{1}{\Omega(2^{-N})} \ll \frac{2^{\beta(N-s)}}{\Omega(2^{-s})}, \quad N \ge s,$$

and

$$J_1 \ll \sum_{s=0}^{+\infty}\left(\frac{\|A_s(f)\|_p}{\Omega(2^{-s})}\right)^\theta \sum_{N=s}^{+\infty} 2^{(s-N)(l-\delta-\beta)\theta} \ll \sum_{s=0}^{+\infty}\left(\frac{\|A_s(f)\|_p}{\Omega(2^{-s})}\right)^\theta. \tag{3.8}$$

Similarly, we choose γ such that $0 < \gamma < \alpha$. Then

$$\begin{aligned} J_2 &= \sum_{N=0}^{+\infty}\left(\frac{1}{\Omega(2^{-N})}\right)^\theta \left(\sum_{s=N+1}^{+\infty} 2^{(N-s)\gamma}2^{-(N-s)\gamma}\|A_s(f)\|_p\right)^\theta \\ &\ll \sum_{N=0}^{+\infty}\left(\frac{1}{\Omega(2^{-N})}\right)^\theta \sum_{s=N+1}^{+\infty} 2^{(s-N)\gamma\theta}\|A_s(f)\|_p^\theta \\ &= \sum_{s=1}^{+\infty}\|A_s(f)\|_p^\theta \sum_{N=0}^{s-1}\left(\frac{2^{(s-N)\gamma}}{\Omega(2^{-N})}\right)^\theta. \end{aligned} \tag{3.9}$$

Since $\Omega(t)/t^{\alpha}$ is almost increasing, we get

$$\frac{1}{\Omega(2^{-N})} \ll \frac{2^{(N-s)\alpha}}{\Omega(2^{-s})}, \quad N \le s. \tag{3.10}$$

It follows from (3.9) and (3.10) that

$$J_2 \ll \sum_{s=1}^{+\infty} \left(\frac{\|A_s(f)\|_p}{\Omega(2^{-s})} \right)^{\theta} \sum_{N=0}^{s-1} 2^{(s-N)(\gamma-\alpha)\theta} \ll \sum_{s=0}^{+\infty} \left(\frac{\|A_s(f)\|_p}{\Omega(2^{-s})} \right)^{\theta}. \tag{3.11}$$

Next, we estimate I_2. Note that $\Omega(t)/t^{\alpha}$ is almost increasing and that $\omega_l(f,t)_p \ll \|f\|_p$. Then

$$I_2 = \int_1^{+\infty} \left(\frac{\omega_l(f,t)_p}{\Omega(t)} \right)^{\theta} \frac{dt}{t} \ll \int_1^{+\infty} \left(\frac{\|f\|_p}{\Omega(t)} \right)^{\theta} \frac{dt}{t} \ll \int_1^{+\infty} \frac{\|f\|_p^{\theta}}{t^{\alpha\theta+1}} dt \ll \|f\|_p^{\theta},$$

and

$$\begin{aligned} \|f\|_p^{\theta} &\le \left(\sum_{s=0}^{+\infty} \|A_s(f)\|_p \right)^{\theta} \ll \left(\sum_{s=0}^{+\infty} (\Omega(2^{-s}))^{\theta'} \right)^{\frac{\theta}{\theta'}} \sum_{s=0}^{+\infty} \left(\frac{\|A_s(f)\|_p}{\Omega(2^{-s})} \right)^{\theta} \\ &\ll \left(\sum_{s=0}^{+\infty} 2^{-s\alpha\theta'} \right)^{\frac{\theta}{\theta'}} \sum_{s=0}^{+\infty} \left(\frac{\|A_s(f)\|_p}{\Omega(2^{-s})} \right)^{\theta} \ll \sum_{s=0}^{+\infty} \left(\frac{\|A_s(f)\|_p}{\Omega(2^{-s})} \right)^{\theta}, \end{aligned}$$

where in the second inequality we used the Hölder inequality, and in the third inequality we used (3.10) with $N = 0$. This together with (3.3), (3.7), (3.8) and (3.11) shows (3.1) for $1 \le \theta < \infty$. For $\theta = \infty$, the proof is similar and simpler; we omit it.

(3°) ⇒ (2°). It follows from the definition of $A_s(f)$ and the properties of the operators V_N that

$$\|A_s(f)\|_p \ll \|f\|_p, \quad \text{and} \quad \|A_s(f)\|_p \ll E_{2^{s-2}}(f)_p.$$

Using (3.2), we get

$$\begin{aligned} \|f\|^{(1)}_{B^{\Omega}_{p,\theta}} &\ll \|f\|_p + \left\{ \sum_{s=2}^{\infty} \left(\frac{E_{2^{s-2}}(f)_p}{\Omega(2^{-s})} \right)^{\theta} \right\}^{\frac{1}{\theta}} \ll \|f\|_p + \left\{ \sum_{s=2}^{\infty} \left(\frac{E_{2^{s-2}}(f)_p}{\Omega(2^{-(s-2)})} \right)^{\theta} \right\}^{\frac{1}{\theta}} \\ &= \|f\|^{(2)}_{B^{\Omega}_{p,\theta}} < +\infty. \end{aligned}$$

(1°) ⇒ (3°). Using (2.1) with $\beta = 0$ and the properties of $\omega_l(f,t)_p$, we get

$$E_{2^s}(f)_p \ll \omega_l(f, 2^{-s})_p \ll \omega_l(f,t)_p, \quad t \in (2^{-s-1}, 2^{-s}).$$

Hence, by (3.2)

$$\frac{E_{2^s}(f)_p}{\Omega(2^{-s})} \ll \frac{\omega_l(f,t)_p}{\Omega(t)}, \quad t \in (2^{-s-1}, 2^{-s}).$$

Then,

$$\begin{aligned} \left\{ \sum_{s=0}^{+\infty} \left(\frac{E_{2^s}(f)_p}{\Omega(2^{-s})} \right)^{\theta} \right\}^{1/\theta} &\ll \left\{ \sum_{s=0}^{+\infty} \int_{2^{-s-1}}^{2^{-s}} \left(\frac{\omega_l(f,t)_p}{\Omega(t)} \right)^{\theta} \frac{dt}{t} \right\}^{1/\theta} \\ &= \left\{ \int_0^1 \left(\frac{\omega_l(f,t)_p}{\Omega(t)} \right)^{\theta} \frac{dt}{t} \right\}^{1/\theta} \le \|f\|_{b^{\Omega}_{p,\theta}} < +\infty. \end{aligned}$$

This means (3°) for $1 \le \theta \le \infty$. Theorem 2 is now proved. □

Remark 1. If the sequence $\{a_n\}_{n=1}^{\infty}$ of positive numbers satisfies $a_n \asymp a_{n_0}$ for all $n \in [n_0, 2n_0]$, then

$$\sum_{n=1}^{\infty} a_n \asymp \sum_{s=0}^{\infty} 2^s a_{2^s}.$$

From this we know that

$$\|f\|_{B^{\Omega}_{p,\theta}} \asymp \|f\|^{(1)}_{B^{\Omega}_{p,\theta}} \asymp \|f\|^{(2)}_{B^{\Omega}_{p,\theta}} \asymp \|f\|^{(3)}_{B^{\Omega}_{p,\theta}} := \begin{cases} \|f\|_p + \left\{ \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{E_n(f)_p}{\Omega(n^{-1})} \right)^{\theta} \right\}^{\frac{1}{\theta}}, & 1 \le \theta < \infty; \\ \|f\|_p + \sup_{n \ge 1} \frac{E_n(f)_p}{\Omega(n^{-1})}, & \theta = \infty. \end{cases}$$

Remark 2. If a series $\sum_{s=0}^{\infty} g_s(f)$ converges to f in the L_p norm and $g_s(f),\ s = 0, 1, \ldots,$ satisfy the Bernstein inequality, then from the proof of Theorem 2, we know that

$$\|f\|_{B^{\Omega}_{p,\theta}} \ll \left\{ \sum_{s=0}^{\infty} \left(\frac{\|g_s(f)\|_p}{\Omega(2^{-s})} \right)^{\theta} \right\}^{\frac{1}{\theta}}.$$

On the other hand, if a sequence $h_s(f),\ s = 0, 1, \ldots,$ satisfies $\|h_s(f)\|_p \ll E_{2^{s-2}}(f)_p$, then

$$\left\{ \sum_{s=0}^{\infty} \left(\frac{\|h_s(f)\|_p}{\Omega(2^{-s})} \right)^{\theta} \right\}^{\frac{1}{\theta}} \ll \|f\|_{B^{\Omega}_{p,\theta}}.$$

If $\Omega(t) \in \Phi^*_l$, then $\Omega(t) \in \Phi^*_{l'}$ for any $l' > l$. From the representation theorem for $B^{\Omega}_{p,\theta}$, we know the following:

Corollary 1. *If $f \in B^{\Omega}_{p\theta}$, $1 \le p, \theta \le \infty$, $\Omega(t) \in \Phi^*_l$, then for every $l' > l$ we have*

$$\|f\|_p + \left\{ \int_0^{\infty} \left(\frac{\omega_l(f,t)_p}{\Omega(t)} \right)^{\theta} \frac{dt}{t} \right\}^{\frac{1}{\theta}} \asymp \|f\|_p + \left\{ \int_0^{\infty} \left(\frac{\omega_{l'}(f,t)_p}{\Omega(t)} \right)^{\theta} \frac{dt}{t} \right\}^{\frac{1}{\theta}},$$

with the usual change for $\theta = \infty$.

The following theorem describes the lifting property of the operator $(-\triangle)^{r/2},\ r > 0$. (See [30, p. 88].)

Theorem 3. *Suppose $\Omega(t) = t^r \Omega_1(t),\ \Omega(t),\ \Omega_1(t) \in \Phi^*_l,\ r > 0$. If $f \in B^{\Omega}_{p,\theta}$, then $(-\triangle)^{r/2} f \in B^{\Omega_1}_{p,\theta}$. Conversely, if $f^{(r)} = (-\triangle)^{r/2} f \in B^{\Omega_1}_{p,\theta}$, then $f \in B^{\Omega}_{p,\theta}$.*

Proof. It follows from the definitions of the operators $(-\triangle)^{r/2}$ and A_s that for any $f \in L_p$,

$$A_s((-\triangle)^{r/2} f) = (-\triangle)^{r/2}(A_s(f)) = (A_s(f))^{(r)}.$$

Using the Bernstein inequality, we get

$$\frac{\|A_s((-\triangle)^{r/2} f)\|_p}{\Omega_1(2^{-s})} = \frac{\|(A_s(f))^{(r)}\|_p}{\Omega_1(2^{-s})} \ll \frac{2^{rs}\|A_s(f)\|_p}{\Omega_1(2^{-s})} = \frac{\|A_s(f)\|_p}{\Omega(2^{-s})}.$$

If $f \in B^{\Omega}_{p,\theta}$, then we have

$$\|(-\triangle)^{r/2} f\|_{B^{\Omega_1}_{p,\theta}} \asymp \left\{ \sum_{s \ge 0} \left(\frac{\|A_s((-\triangle)^{r/2} f)\|_p}{\Omega_1(2^{-s})} \right)^{\theta} \right\}^{\frac{1}{\theta}} \ll \left\{ \sum_{s \ge 0} \left(\frac{\|A_s(f)\|_p}{\Omega(2^{-s})} \right)^{\theta} \right\}^{\frac{1}{\theta}} < +\infty,$$

and therefore, $(-\triangle)^{r/2}f \in B^{\Omega_1}_{p,\theta}$. On the other hand, if $f^{(r)} \in B^{\Omega_1}_{p,\theta}$, then by (2.1) we get

$$\frac{E_{2^s}(f)_p}{\Omega(2^{-s})} \ll \frac{2^{-sr}\omega_l(f^{(r)}, 2^{-s})_p}{\Omega(2^{-s})} \ll \frac{\omega_l(f^{(r)}, t)_p}{\Omega_1(t)}, \quad t \in (2^{-s-1}, 2^{-s}).$$

Hence,

$$\left\{\sum_{s=0}^{+\infty}\left(\frac{E_{2^s}(f)_p}{\Omega(2^{-s})}\right)^{\theta}\right\}^{1/\theta} \ll \left\{\sum_{s=0}^{+\infty}\int_{2^{-s-1}}^{2^{-s}}\left(\frac{\omega_l(f^{(r)}, t)_p}{\Omega_1(t)}\right)^{\theta}\frac{dt}{t}\right\}^{1/\theta}$$
$$= \left\{\int_0^1\left(\frac{\omega_l(f^{(r)}, t)_p}{\Omega_1(t)}\right)^{\theta}\frac{dt}{t}\right\}^{1/\theta} \le \|f^{(r)}\|_{b^{\Omega_1}_{p,\theta}} < +\infty,$$

and therefore, $f \in B^{\Omega}_{p,\theta}$. Theorem 3 is proved. □

Theorem 4 (*Embedding Theorem*)**.** *Suppose that $\Omega(t) = t^r\Omega_1(t)$, $\Omega(t), \Omega_1(t) \in \Phi_l^*$, $1 \le p < q \le \infty$, $1 \le \theta, \theta_1 \le \infty$. If $f \in B^{\Omega}_{p,\theta}$ and $r > \left(\frac{1}{p}-\frac{1}{q}\right)(d-1)$, then $f \in B^{\Omega_1}_{q,\theta_1}$. Moreover, If $f \in B^{\Omega}_{p,\theta}$ and $r \ge \left(\frac{1}{p}-\frac{1}{q}\right)(d-1)$, then $f \in L_q$.*

Proof. Using the Nikolskii inequality (see [7]), we get for $1 \le p < q \le \infty$

$$\|A_s(f)\|_q \ll 2^{s(d-1)(1/p-1/q)}\|A_s(f)\|_p.$$

It follows from Theorem 2 and the Hölder inequality that

$$\|f\|_{B^{\Omega_1}_{q,\theta_1}} \ll \left\{\sum_{s\ge 0}\left(\frac{\|A_s(f)\|_q}{\Omega_1(2^{-s})}\right)^{\theta_1}\right\}^{\frac{1}{\theta_1}} \le \sum_{s\ge 0}\frac{\|A_s(f)\|_q}{\Omega_1(2^{-s})}$$
$$\le \sum_{s\ge 0}\frac{\|A_s(f)\|_p}{\Omega(2^{-s})}2^{s((d-1)(1/p-1/q)-r)}$$
$$\le \left\{\sum_{s\ge 0}\left(\frac{\|A_s(f)\|_p}{\Omega(2^{-s})}\right)^{\theta}\right\}^{\frac{1}{\theta}}\left\{\sum_{s\ge 0}2^{s\theta'((d-1)(1/p-1/q)-r)}\right\}^{\frac{1}{\theta'}}$$
$$\ll \left\{\sum_{s\ge 0}\left(\frac{\|A_s(f)\|_p}{\Omega(2^{-s})}\right)^{\theta}\right\}^{\frac{1}{\theta}} < +\infty,$$

where $1/\theta + 1/\theta' = 1$. Similarly, we have

$$\|f\|_q \le \sum_{s\ge 0}\|A_s(f)\|_q \ll \sum_{s\ge 0}\frac{\|A_s(f)\|_p}{\Omega(2^{-s})}\Omega(2^{-s})2^{s(d-1)(1/p-1/q)}$$
$$\le \sum_{s\ge 0}\frac{\|A_s(f)\|_p}{\Omega(2^{-s})}\Omega_1(2^{-s}) \le \left\{\sum_{s\ge 0}\left(\frac{\|A_s(f)\|_p}{\Omega(2^{-s})}\right)^{\theta}\right\}^{\frac{1}{\theta}}\left\{\sum_{s\ge 0}(\Omega_1(2^{-s}))^{\theta'}\right\}^{\frac{1}{\theta'}}$$
$$\ll \left\{\sum_{s\ge 0}\left(\frac{\|A_s(f)\|_p}{\Omega(2^{-s})}\right)^{\theta}\right\}^{\frac{1}{\theta}} < +\infty.$$

This completes the proof of Theorem 4. □

Lemma 1. *Suppose that* $\Omega(t) = t^r\Omega_1(t), \Omega(t), \Omega_1(t) \in \Phi_l^*, 1 \le p, q, \theta \le \infty, r \ge (d-1)\left(\frac{1}{p}-\frac{1}{q}\right)_+$. *Then*

$$\sup_{f\in BB^{\Omega}_{p,\theta}} E_N(f)_q \asymp \sup_{f\in BB^{\Omega}_{p,\theta}} \|f - V_N(f)\|_q \ll \Omega(N^{-1})N^{(d-1)\left(\frac{1}{p}-\frac{1}{q}\right)_+}.$$

Proof. It follows from the Nikolskii inequality that for $1 \le p, q \le \infty$,

$$\|A_s(f)\|_q \ll 2^{s(d-1)\left(\frac{1}{p}-\frac{1}{q}\right)_+}\|A_s(f)\|_p.$$

Choose $n \in \mathbb{N}$ such that $2^n \le N < 2^{n+1}$. Then for any $f \in BB^{\Omega}_{p,\theta}$, we have

$$\begin{aligned}
\|f - V_N(f)\|_q &\ll E_N(f)_q \ll E_{2^n}(f)_q \ll \|f - V_{2^{n-1}}(f)\|_q \\
&\ll \sum_{s=n+1}^{\infty}\|A_s(f)\|_q \ll \sum_{s=n+1}^{\infty} 2^{s(d-1)\left(\frac{1}{p}-\frac{1}{q}\right)_+}\|A_s(f)\|_p \\
&\ll \left(\sum_{s=n+1}^{\infty}\left(\Omega(2^{-s})2^{s(d-1)\left(\frac{1}{p}-\frac{1}{q}\right)_+}\right)^{\theta'}\right)^{\frac{1}{\theta'}}\left(\sum_{s=n+1}^{\infty}\left(\frac{\|A_s(f)\|_p}{\Omega(2^{-s})}\right)^{\theta}\right)^{\frac{1}{\theta}} \\
&\ll \left(\sum_{s=n+1}^{\infty}\left(\Omega_1(2^{-s})2^{-s\left(r-(d-1)\left(\frac{1}{p}-\frac{1}{q}\right)_+\right)}\right)^{\theta'}\right)^{\frac{1}{\theta'}}\|f\|_{B^{\Omega}_{p,\theta}} \\
&\ll 2^{-n\left(r-(d-1)\left(\frac{1}{p}-\frac{1}{q}\right)_+\right)}\left(\sum_{s=n+1}^{\infty}\left(\Omega_1(2^{-s})\right)^{\theta'}\right)^{\frac{1}{\theta'}} \\
&\ll \Omega_1(2^{-n})2^{-nr}2^{n(d-1)\left(\frac{1}{p}-\frac{1}{q}\right)_+} \ll \Omega(N^{-1})N^{(d-1)\left(\frac{1}{p}-\frac{1}{q}\right)_+},
\end{aligned}$$

where $1/\theta + 1/\theta' = 1$. This completes the proof of Lemma 1. □

4. Characterization of $B^{\Omega}_{p,\theta}(\mathbb{S}^{d-1})$ using a frame

In this section, we give the characterization of $B^{\Omega}_{p,\theta}(\mathbb{S}^{d-1})$ using a frame. We follow the approach of Dai [8] and Narcowich et al. [23,24] and their construction of "a frame" (or "a needlet"). Similar polynomial frames on the interval can be seen in [20]. Using the frame, we give the characterization of $B^{\Omega}_{p,\theta}(\mathbb{S}^{d-1})$ in terms of the coefficients in the frame decompositions.

Denote by $d(x, y)$ the geodesic distance $\arccos x \cdot y$ between two points x and y on $\mathbb{S}^{d-1}$ and $B(x, r)$ the ball centered at $x \in \mathbb{S}^{d-1}$ and having radius $r > 0$, i.e., $B(x, r) = \{y \in \mathbb{S}^{d-1} : d(x, y) \le r\}$, by $\#E$ the number of the elements in E, and by $|E|$ the measure $\sigma(E)$ of a measurable subset $E \subset \mathbb{S}^{d-1}$. It is well known that for any $x \in \mathbb{S}^{d-1}$ and any $r \in (0, \pi)$, $|B(x, r)| \asymp r^{d-1}$. The construction of the frame is based on the following positive cubature formulae and Marcinkiewicz–Zygmund inequalities (see [7,5,19,24]).

Theorem A. *There exists a constant* $\gamma > 0$ *depending only on d such that for any integer* $N > 0$ *and any finite set* $\{\xi_k\}_{k\in\Lambda}$ *of distinct points* $\xi_k \in \mathbb{S}^{d-1}$ *satisfying*

$$\min_{\substack{i,j\in\Lambda\\ i\ne j}} d(\xi_i, \xi_j) \ge \frac{\gamma}{N}, \quad \text{and} \quad \max_{x\in\mathbb{S}^{d-1}}\min_{j\in\Lambda} d(x, \xi_j) < \frac{\gamma}{N},$$

there exists a set of numbers $a_{N,k} \asymp N^{-(d-1)}$, $k \in \Lambda$, *such that for any* $f \in \Pi^d_N$, $1 \le p \le \infty$,

$$\int_{\mathbb{S}^{d-1}} f(y)d\sigma(y) = \sum_{k\in\Lambda} a_{N,k}f(\xi_k),$$

and

$$\|f\|_p \asymp \begin{cases} \left(\dfrac{1}{N^{d-1}} \displaystyle\sum_{k\in\Lambda} |f(x_{j,k})|^p\right)^{\frac{1}{p}}, & \textit{if } 1 \le p < \infty, \\ \max\limits_{k\in\Lambda}(|f(x_{j,k})|), & \textit{if } p = \infty, \end{cases}$$

where the constants of equivalence depend only on d.

Now to each integer $j > 0$ we assign a finite set $\{x_{j,k} : k \in \Lambda_j^d\}$ of distinct points $x_{j,k} \in \mathbb{S}^{d-1}$ satisfying

$$\min_{\substack{k,k'\in\Lambda_j^d \\ k\neq k'}} d(x_{j,k}, x_{j,k'}) \ge \frac{\gamma}{2^{j+4}} \quad \text{and} \quad \max_{x\in\mathbb{S}^{d-1}} \min_{k\in\Lambda_j^d} d(x, x_{j,k}) < \frac{\gamma}{2^{j+4}},$$

with γ as in Theorem A. Evidently, $\#\Lambda_j^d \asymp 2^{j(d-1)}$. By Theorem A, there exists a set of numbers $\lambda_{j,k} \asymp 2^{-j(d-1)}$, $k \in \Lambda_j^d$, such that for any $1 \le p \le \infty$ and any $f \in \Pi_{2^{j+4}}^d$,

$$\int_{\mathbb{S}^{d-1}} f(y)d\sigma(y) = \sum_{k\in\Lambda_j^d} \lambda_{j,k} f(x_{j,k}), \tag{4.1}$$

$$\|f\|_p \asymp \begin{cases} \left(\dfrac{1}{2^{j(d-1)}} \displaystyle\sum_{k\in\Lambda_j^d} |f(x_{j,k})|^p\right)^{\frac{1}{p}}, & \text{if } 1 \le p < \infty, \\ \max\limits_{k\in\Lambda_j^d}(|f(x_{j,k})|), & \text{if } p = \infty, \end{cases} \tag{4.2}$$

where the constants of equivalence depend only on d. For convenience, we also set $\Lambda_0^d = \{0\}$, $\lambda_{0,0} = 1$ and take $x_{0,0}$ to be any fixed point on $\mathbb{S}^{d-1}$.

Let ϕ be a nonnegative C^∞-function on $\mathbb{R}$ supported in $\{x \in \mathbb{R} : \frac{1}{2} \le |x| \le 2\}$ and satisfying

$$\sum_{j=-\infty}^{\infty} \left(\phi(2^{-j}x)\right)^2 = 1, \quad \text{for all } x \neq 0.$$

Such a ϕ exists (see [23]). Let P_k, $k \in \mathbb{N}$, be the polynomials defined by

$$P_k(t) := \frac{2k+d-2}{d-2} P_k^{\frac{d-2}{2}}(t), \quad t \in [-1, 1],$$

where $P_k^{\frac{d-2}{2}}(t)$ denotes the usual ultraspherical polynomial of order $\frac{d-2}{2}$ normalized by $P_k^{\frac{d-2}{2}}(1) = \frac{\Gamma(k+d-2)}{\Gamma(d-2)\Gamma(k+1)}$. (For a precise definition of ultraspherical polynomials, we refer the reader to [28, p. 81]). We define, together with ϕ, a sequence of functions (the "frame" on the sphere)

$$\psi_{j,k}(x) := \sqrt{\lambda_{j,k}} G_j(x \cdot x_{j,k}), \quad j \ge 0,\ k \in \Lambda_j^d,$$

where

$$G_0(t) = 1, \qquad G_j(t) = \sum_{k=[2^{j-2}]}^{2^j} \phi\left(\frac{k}{2^{j-1}}\right) P_k(t), \quad t \in [-1, 1],\ j \ge 1. \tag{4.3}$$

It is well known that for $f \in L(\mathbb{S}^{d-1})$,

$$H_k^d(f)(x) = \int_{\mathbb{S}^{d-1}} f(y) P_k(x \cdot y) d\sigma(y), \quad x \in \mathbb{S}^{d-1},$$

where $H_k^d(f)$ denotes the orthogonal projection of f onto $\mathcal{H}_k^d$. It follows that

$$\langle f, G_j(x\cdot)\rangle = \sum_{k=[2^{j-2}]}^{2^j} \phi\left(\frac{k}{2^{j-1}}\right) H_k^d(f)(x) \quad \text{and}$$

$$\sum_{k\in\Lambda_j^d} \langle P, \psi_{j,k}\rangle \psi_{j,k}(x) = \sum_{k=[2^{j-2}]}^{2^j} \phi^2\left(\frac{k}{2^{j-1}}\right) H_k^d(f)(x).$$

Thus, associated with each distribution f on $\mathbb{S}^{d-1}$, there is a series $\sum_{j=0}^{\infty}\sum_{k\in\Lambda_j^d}\langle f, \psi_{j,k}\rangle\psi_{j,k}$, and moreover, for each spherical polynomial P, we have

$$P(x) = \sum_{j=0}^{\infty}\sum_{k\in\Lambda_j^d}\langle P, \psi_{j,k}\rangle\psi_{j,k}(x)$$

with only a finite number of nonzero coefficients $\langle P, \psi_{j,k}\rangle$, as can be easily verified. We will keep the above notation for the rest of the paper.

Theorem 5. *Let $\Omega(t) \in \Phi_l^*,\ l > 0,\ 1 \le p, \theta \le \infty$. Then for $f \in B_{p,\theta}^{\Omega}$, we have*

$$\|f\|_{B_{p,\theta}^{\Omega}} \asymp \left(\sum_{j=0}^{\infty}\left(\frac{2^{-j(d-1)\left(\frac{1}{p}-\frac{1}{2}\right)}}{\Omega(2^{-j})}\right)^{\theta}\left(\sum_{k\in\Lambda_j^d}|\langle f, \psi_{j,k}\rangle|^p\right)^{\frac{\theta}{p}}\right)^{\frac{1}{\theta}}, \tag{4.4}$$

with the usual change when $p = \infty$ and/or $\theta = \infty$, where the constant of equivalence is independent of f. In addition, if $\{a_{j,k} : j = 1, 2, \ldots,\ k \in \Lambda_j^d\}$ is a sequence of real numbers such that

$$\left(\sum_{j=0}^{\infty}\left(\frac{2^{-j(d-1)\left(\frac{1}{p}-\frac{1}{2}\right)}}{\Omega(2^{-j})}\right)^{\theta}\left(\sum_{k\in\Lambda_j^d}|a_{j,k}|^p\right)^{\frac{\theta}{p}}\right)^{\frac{1}{\theta}} < \infty, \tag{4.5}$$

then the series $\sum_{j=0}^{\infty}\sum_{k\in\Lambda_j^d} a_{j,k}\psi_{j,k}$ converges unconditionally to some $f \in B_{p,\theta}^{\Omega}$ on $\mathbb{S}^{d-1}$, and moreover,

$$\|f\|_{B_{p,\theta}^{\Omega}} \ll \left(\sum_{j=0}^{\infty}\left(\frac{2^{-j(d-1)\left(\frac{1}{p}-\frac{1}{2}\right)}}{\Omega(2^{-j})}\right)^{\theta}\left(\sum_{k\in\Lambda_j^d}|a_{j,k}|^p\right)^{\frac{\theta}{p}}\right)^{\frac{1}{\theta}}, \tag{4.6}$$

with the usual change when $p = \infty$ and/or $\theta = \infty$.

Proof. We define, for $f \in \mathcal{S}'(\mathbb{S}^{d-1})$,

$$\sigma_j(f)(x) := \langle f, G_j(x\cdot)\rangle, \quad x \in \mathbb{S}^{d-1},\ j = 0, 1, 2, \ldots,$$

where G_j is defined by (4.3). First we claim that for $f \in B_{p,\theta}^{\Omega}$,

$$\|f\|_{B_{p,\theta}^{\Omega}} \asymp \left\{\sum_{j=0}^{\infty}\left(\frac{\|\sigma_j(f)\|_p}{\Omega(2^{-j})}\right)^{\theta}\right\}^{\frac{1}{\theta}}. \tag{4.7}$$

In fact, from the definition we know that the operators $\sigma_j,\ j \in \mathbb{Z}_+$, are uniformly bounded on L_p. By Lemma 2.1 in [8], we get that for $f \in L_p$, the series $\sum_{j=0}^{\infty}\sigma_j \circ \sigma_j(f)$ converges to f in the space L_p. Since for each $k \in \mathbb{Z}_+$, $\sum_{j=0}^{k} \sigma_j \circ \sigma_j(f) \in \Pi_{2^k}^d$, from Remark 2 we get

$$\|f\|_{B_{p,\theta}^{\Omega}} \ll \left\{ \sum_{j=0}^{\infty} \left(\frac{\|\sigma_j \circ \sigma_j(f)\|_p}{\Omega(2^{-j})} \right)^{\theta} \right\}^{\frac{1}{\theta}} \ll \left\{ \sum_{j=0}^{\infty} \left(\frac{\|\sigma_j(f)\|_p}{\Omega(2^{-j})} \right)^{\theta} \right\}^{\frac{1}{\theta}}.$$

On the other hand, noting that $\sigma_j(g) = 0$ for any $g \in \Pi_{2^{j-2}}^d$ and $j \geq 2$, we obtain that for $j \geq 2$,

$$\|\sigma_j(f)\|_p = \inf_{g \in \Pi_{2^{j-2}}^d} \|\sigma_j(f-g)\|_p \ll E_{2^{j-2}}(f)_p.$$

This, combined with Remark 2, gives the lower estimates of $\|f\|_{B_{p,\theta}^{\Omega}}$ and completes the proof of (4.7).

Next we show (4.4). Note that for each $j \geq 0$, $\sigma_j(f) \in \Pi_{2^j}^d$ and $\langle f, \psi_{j,k} \rangle = \sqrt{\lambda_{j,k}} \sigma_j(f)(x_{j,k})$. Thus, it follows from (4.1) and (4.2) that for $j \geq 0$ and $1 \leq p \leq \infty$,

$$\|\sigma_j(f)\|_p \asymp 2^{-j(d-1)\left(\frac{1}{p}-\frac{1}{2}\right)} \left(\sum_{k \in \Lambda_j^d} |\langle f, \psi_{j,k} \rangle|^p \right)^{\frac{1}{p}}, \tag{4.8}$$

with the usual change when $p = \infty$. This together with (4.7) implies (4.4) for $1 \leq p \leq \infty$.

Finally, we show (4.6). Without loss of generality, we may assume that only a finite number of the coefficients $a_{j,k}$ are nonzero. Then, $f = \sum_{j=0}^{\infty} \sum_{k \in \Lambda_j^d} a_{j,k} \psi_{j,k}$ is a spherical polynomial and $f \in B_{p,\theta}^{\Omega}$. Note that $\sum_{k \in \Lambda_j^d} a_{j,k} \psi_{j,k} := g_j(f) \in \Pi_{2^j}^d$. Then by Remark 2, we have

$$\|f\|_{B_{p,\theta}^{\Omega}} \ll \left\{ \sum_{j=0}^{\infty} \left(\frac{\|g_j(f)\|_p}{\Omega(2^{-j})} \right)^{\theta} \right\}^{\frac{1}{\theta}}.$$

Obviously, $g_j(f)$ can be viewed as a linear operator T_j from $\mathbb{R}^{\#\Lambda_j^d}$ to $\Pi_{2^j}^d$ through

$$T_j a(x) := \sum_{k \in \Lambda_j^d} a_{j,k} \psi_{j,k}(x) = g_j(f),$$

where $a := (a_{j,k})_{k \in \Lambda_j^d} \in R^{\#\Lambda_j^d}$. It suffices to show that for $1 \leq p \leq \infty$,

$$\|T_j(a)\|_p = \left\| \sum_{k \in \Lambda_j^d} a_{j,k} \psi_{j,k} \right\|_p \ll 2^{j(d-1)\left(\frac{1}{2}-\frac{1}{p}\right)} \left(\sum_{k \in \Lambda_j^d} |a_{j,k}|^p \right)^{\frac{1}{p}}. \tag{4.9}$$

For $p = 1$, we have

$$\|T_j a\|_1 \leq \sum_{k \in \Lambda_j^d} |a_{j,k}| \|\psi_{j,k}\|_1 \ll 2^{-\frac{j(d-1)}{2}} \sum_{k \in \Lambda_j^d} |a_{j,k}|,$$

where here, we used the fact $\|\psi_{j,k}\|_p \asymp 2^{-j(1/p-1/2)(d-1)}$; for $p = \infty$, by Formula 5.6 in [8], we have

$$\|Ta\|_\infty \ll 2^{j(d-1)/2} \left(\max_{k \in \Lambda_j^d} |a_{j,k}| \right);$$

and for $1 < p < \infty$, (4.9) follows from the Riesz–Thorin theorem. This completes the proof of Theorem 5. $\square$

5. Discretization of the problem of estimates of widths

Let $\Lambda^d = (\Lambda_j^d)_{j\in\mathbb{N}}$ denote the sequence of the sets $\Lambda_j^d, j \in \mathbb{N}$, where Λ_j^d are the sets defined in Section 3. We define the space $b_{p,\theta}^{\Omega,\Lambda^d}$ of sequences $a = \{a_{j,k}\}_{j\in\mathbb{N},k\in\Lambda_j^d}$ with finite norm

$$\|a\|_{b_{p,\theta}^{\Omega,\Lambda^d}} = \left(\sum_{j=0}^{\infty}\left(\frac{2^{j(d-1)\left(\frac{1}{2}-\frac{1}{p}\right)}}{\Omega(2^{-j})}\right)^{\theta}\left(\sum_{k\in\Lambda_j^d}|a_{j,k}|^p\right)^{\frac{\theta}{p}}\right)^{\frac{1}{\theta}},$$

with the usual modification if $p = \infty$ and/or $\theta = \infty$.

For fixed $j \in \mathbb{N}$, let $\ell_p^{\#\Lambda_j^d}$ $(1 \le p \le \infty)$ denote the space $\mathbb{R}^{\#\Lambda_j^d}$ equipped with the norm

$$\|a\|_{\ell_p^{\#\Lambda_j^d}} := \begin{cases} \left(\sum_{k\in\Lambda_j^d}|a_{j,k}|^p\right)^{\frac{1}{p}}, & 1 \le p < \infty; \\ \max_{k\in\Lambda_j^d}|a_{j,k}|, & p = \infty. \end{cases}$$

The unit ball of $\ell_p^{\#\Lambda_j^d}$ is denoted by $B\ell_p^{\#\Lambda_j^d}$.

Lemma 2. *Let $\Omega(t) \in \Phi_l^*,\ l > 0,\ 1 \le p, q, \theta \le \infty$. Then for $1 \le p \le q \le \infty$, we have*

$$S_n\left(BB_{p,\theta}^{\Omega}(\mathbb{S}^{d-1}),\ L_q(\mathbb{S}^{d-1})\right) \ll \sum_{j=0}^{\infty}\Omega(2^{-j})2^{j(d-1)\left(\frac{1}{p}-\frac{1}{q}\right)}S_{n_j}(B\ell_p^{\#\Lambda_j^d}, \ell_q^{\#\Lambda_j^d}),$$

where S_n denotes one of the symbols d_n, δ_n and d^n, and $\sum_{j=0}^{\infty} n_j \le n$.

Proof. Let X and Y be the normed linear space and let T be the bounded linear operator from X to Y (we write $T \in B(X, Y)$). We define

$$S_n(T) = S_n(T(BX), Y),$$

where BX is the unit ball of X, and S_n denotes one of the symbols d_n, δ_n and d^n. It is well known that $S_n(T)$ has following four properties (see [17,25]):

(i) If rank $T < n$, then $S_n(T) = 0$.
(ii) $\|T\| = S_0(T) \ge S_1(T) \ge S_2(T) \ge \cdots \ge 0$.
(iii) $S_n(T_1TT_2) \le \|T_1\|\,S_n(T)\,\|T_2\|$ for all $T_2 \in B(W, X),\ T \in B(X, Y),\ T_1 \in B(Y, Z)$ and all $n \in \mathbb{N}$.
(iv) $S_{n+m}(T_1 + T_2) \le S_m(T_1) + S_n(T_2)$ for all $T_1, T_2 \in B(X, Y)$ and $n \in \mathbb{N}$.

We will use the above properties below. First we define the linear operator $U : B_{p,\theta}^{\Omega} \longmapsto b_{p,\theta}^{\Omega,\Lambda^d}$ as follows:

$$Uf = \{< f, \psi_{j,k} >\}_{j\in\mathbb{N},k\in\Lambda_j^d} \in b_{p,\theta}^{\Omega,\Lambda^d}, \quad f \in B_{p,\theta}^{\Omega}.$$

It follows from (4.4) that

$$\|U\| \ll 1. \tag{5.1}$$

The linear operator $V : b_{q,1}^{1,\Lambda^d} \longmapsto L_q$ is defined by

$$Va = \sum_{j=0}^{\infty}\sum_{k\in\Lambda_j^d} a_{j,k}\psi_{j,k},$$

where $a=\{a_{j,k}\}_{j\in\mathbb{N},k\in\Lambda_j^d}$. Using (4.9) we have

$$\|Va\|_q \le \sum_{j=0}^{\infty}\|\sum_{k\in\Lambda_j^d} a_{j,k}\psi_{j,k}\|_q \ll \sum_{j=0}^{\infty} 2^{j(d-1)\left(\frac{1}{2}-\frac{1}{q}\right)}\left(\sum_{k\in\Lambda_j^d}|a_{j,k}|^q\right)^{\frac{1}{q}},$$

which means

$$\|V\| \ll 1. \tag{5.2}$$

Since $f=\sum_{j=0}^{\infty}\sum_{k\in\Lambda_j^d} < f,\psi_{j,k} > \psi_{j,k} = VU(f)$ for any $f\in B_{p,\theta}^{\Omega}$, we can factor the identity $Id: B_{p,\theta}^{\Omega}\longmapsto L_q$ as follows:

$$B_{p,\theta}^{\Omega}\xrightarrow{U} b_{p,\theta}^{\Omega,\Lambda^d}\xrightarrow{id} b_{q,1}^{1,\Lambda^d}\xrightarrow{V} L_q.$$

It then follows from (5.1) and (5.2) that

$$S_n\left(BB_{p,\theta}^{\Omega}(\mathbb{S}^{d-1}),\ L_q(\mathbb{S}^{d-1})\right) = S_n(Id)\le \|U\|\|V\|S_n(id)\ll S_n(id),\quad n\in\mathbb{N}. \tag{5.3}$$

Next, for every fixed $j\in\mathbb{N}$, we define the mappings P_j, Q_j as follows:

$$\begin{aligned} &P_j: b_{p,\theta}^{\Omega,\Lambda^d}\longmapsto \ell_p^{\#\Lambda_j^d}\\ &\{a_{i,k}\}_{i\in\mathbb{N},k\in\Lambda_i^d}\longmapsto \{a_{j,k}\}_{k\in\Lambda_j^d};\end{aligned}$$

and

$$\begin{aligned} &Q_j: \ell_q^{\#\Lambda_j^d}\longmapsto b_{q,1}^{1,\Lambda^d}\\ &\{a_{j,k}\}_{k\in\Lambda_j^d}\longmapsto \{b_{i,k}\}_{i\in\mathbb{N},k\in\Lambda_i^d},\end{aligned}$$

where $b_{i,k}=a_{j,k}$ if $i=j$, $b_{i,k}=0$ otherwise. Then

$$\|P_j\|\le \frac{\Omega(2^{-j})}{2^{j(d-1)\left(\frac{1}{2}-\frac{1}{p}\right)}},\qquad \|Q_j\|\le 2^{j(d-1)\left(\frac{1}{2}-\frac{1}{q}\right)}. \tag{5.4}$$

For fixed j, we denote by id_j the identity operator from $\ell_p^{\#\Lambda_j^d}$ to $\ell_q^{\#\Lambda_j^d}$. Then we have

$$id=\sum_{j=0}^{\infty} Q_j\, id_j\, P_j. \tag{5.5}$$

It then follows from (5.3)–(5.5) that

$$\begin{aligned} S_n(Id) &\ll S_n(id)\le \sum_{j=0}^{\infty}\|P_j\|\|Q_j\|S_{n_j}(id_j)\\ &\le \sum_{j=0}^{\infty}\Omega(2^{-j})2^{j(d-1)\left(\frac{1}{p}-\frac{1}{q}\right)}S_{n_j}(B\ell_p^{\#\Lambda_j^d},\ \ell_q^{\#\Lambda_j^d}),\end{aligned}$$

where $\sum_{j=0}^{\infty} n_j\le n$. $\square$

Lemma 3. *Let $\Omega(t)\in\Phi_l^*$, $l>0$, $1\le p,q,\theta\le\infty$. Then there exists an N, $N\asymp n$, $N\ge 2n$, such that*

$$S_n\left(BB_{p,\theta}^{\Omega}(\mathbb{S}^{d-1}),\ L_q(\mathbb{S}^{d-1})\right)\gg \Omega\left(n^{-\frac{1}{d-1}}\right)n^{\frac{1}{p}-\frac{1}{q}}S_n(B\ell_p^N,\ \ell_q^N),$$

where S_n denotes one of the symbols d_n, δ_n and d^n.

Proof. We assume that $b_1 m^{d-1} \le n \le b_2 m^{d-1}$ with $b_1, b_2 > 0$ being independent of n and m. We let $\{x_j\}_{j=1}^N \subset \mathbb{S}^{d-1}$ such that $N \asymp m^{d-1}$ and

$$B\left(x_i, \frac{1}{m}\right) \cap B\left(x_j, \frac{1}{m}\right) = \emptyset, \quad \text{if } i \neq j.$$

We take $b_2 > 0$ sufficiently small that $N \ge 2n$. Let φ be a C^∞-function on $\mathbb{R}$ supported in $\left[\frac{1}{2}, 1\right]$ and equal to 1 on $\left[\frac{2}{3}, \frac{3}{4}\right]$. We define

$$\varphi_i(x) = \varphi(md(x, x_i)), \quad 1 \le i \le N,$$

and set

$$A_N = \left\{ f_a(x) = \sum_{j=1}^N a_j \varphi_j(x) \mid a = (a_1, \dots, a_N) \in \mathbb{R}^N \right\}.$$

Clearly,

$$\operatorname{supp} \varphi_i \subset B\left(x_i, \frac{1}{m}\right) \quad \text{and} \quad \operatorname{supp} \varphi_i \cap \operatorname{supp} \varphi_j = \emptyset. \quad (i \neq j).$$

Hence, if $f \in A_N$, it is easy to verify that

$$\|f_a\|_p \asymp m^{-\frac{d-1}{p}} \|a\|_{l_p^N}. \tag{5.6}$$

Denote by P_{A_N} the orthogonal projector onto A_N, that is, for any $f \in L_1$ and $x \in \mathbb{S}^{d-1}$,

$$P_{A_N}(f)(x) = \sum_{j=1}^N \frac{\varphi_j(x)}{\|\varphi_j\|_2^2} \int_{\mathbb{S}^{d-1}} f(y)\varphi_j(y) d\sigma(y).$$

Then P_{A_N} is the bounded projection operator from L_p to $A_N \cap L_p$. In fact,

$$P_{A_N}(f)(x) = \int_{\mathbb{S}^{d-1}} f(y)K(x,y)d\sigma(y), \quad \text{where } K(x,y) = \sum_{j=1}^N \frac{\varphi_j(x)\varphi_j(y)}{\|\varphi_j\|_2^2}.$$

Using (4.6), we get

$$\int_{\mathbb{S}^{d-1}} |K(x,y)| d\sigma(y) \ll \sum_{j=1}^N |\varphi_j(x)| \ll 1 \quad \forall x \in \mathbb{S}^{d-1}.$$

By the Hölder inequality, it follows that

$$\begin{aligned}
\int_{\mathbb{S}^{d-1}} |P_{A_N}(f)(x)|^p \, d\sigma(x) &= \int_{\mathbb{S}^{d-1}} \left| \int_{\mathbb{S}^{d-1}} f(y)K(x,y) \, d\sigma(y) \right|^p \, d\sigma(x) \\
&\le \int_{\mathbb{S}^{d-1}} \left(\int_{\mathbb{S}^{d-1}} |f(y)|^p |K(x,y)| \, d\sigma(y) \right) \cdot \left(\int_{\mathbb{S}^{d-1}} |K(x,y)| \, d\sigma(y) \right)^{\frac{p}{p'}} \, d\sigma(x) \\
&\ll \int_{\mathbb{S}^{d-1}} \left(\int_{\mathbb{S}^{d-1}} |f(y)|^p |K(x,y)| \, d\sigma(y) \right) \, d\sigma(x) \\
&= \int_{\mathbb{S}^{d-1}} \left(\int_{\mathbb{S}^{d-1}} |K(x,y)| \, d\sigma(x) \right) |f(y)|^p \, d\sigma(y) \\
&\ll \int_{\mathbb{S}^{d-1}} |f(y)|^p \, d\sigma(y),
\end{aligned}$$

which implies that

$$\|P_{A_N}(f)\|_p \ll \|f\|_p.$$

Hence, we have

$$S_n\left(BB^{\Omega}_{p,\theta}(\mathbb{S}^{d-1}), L_q(\mathbb{S}^{d-1})\right) \geq S_n\left(BB^{\Omega}_{p,\theta} \cap A_N, L_q\right) \gg S_n\left(BB^{\Omega}_{p,\theta} \cap A_N, L_q \cap A_N\right). \tag{5.7}$$

Next we will prove

$$c\Omega\left(n^{-\frac{1}{d-1}}\right)(A_N \cap BL_p) \subset A_N \cap BB^{\Omega}_{p,\theta}. \tag{5.8}$$

For any $f_a \in A_N$, we have by Theorem 2 and (2.1) that

$$\|f_a\|_{B^{\Omega}_{p,\theta}} \leq \|f_a\|_{B^{\Omega}_{p,1}} \ll \|f_a\|_p + \sum_{s=0}^{+\infty} \frac{E_{2^s}(f_a)_p}{\Omega(2^{-s})} \ll \sum_{s=0}^{J} \frac{\|f_a\|_p}{\Omega(2^{-s})} + \sum_{s=J+1}^{+\infty} \frac{\|f_a^{(l)}\|_p}{2^{sl}\Omega(2^{-s})} := I_1 + I_2, \tag{5.9}$$

where J satisfies $2^J \asymp m$. Since $\Omega(t) \in \Phi_l^*$, we get that there is an $\alpha > 0$ such that $\Omega(t)/t^{\alpha}$ is almost increasing. It follows that

$$\frac{1}{\Omega(2^{-s})} \ll \frac{2^{-(J-s)\alpha}}{\Omega(2^{-J})},$$

and hence

$$I_1 \ll \frac{\|f_a\|_p}{\Omega(2^{-J})} \sum_{s=0}^{J} 2^{-(J-s)\alpha} \ll \frac{\|f_a\|_p}{\Omega(2^{-J})} \ll \frac{\|f_a\|_p}{\Omega\left(n^{-\frac{1}{d-1}}\right)}, \quad (\text{by } 2^J \asymp m \asymp n^{\frac{1}{d-1}}). \tag{5.10}$$

For a positive integer $v > l$ and $f_a \in A_N$, we note that in geodesic polar coordinates of the sphere (the polar point is x_i), the Laplace–Beltrami operator $\triangle$ on $\mathbb{S}^{d-1}$ equals $\triangle_\theta + \triangle'$, where $\triangle'$ denotes the Laplace–Beltrami operator on the sphere $\{x \in \mathbb{S}^{d-1} \mid d(x, x_i) = \theta\}$ in $\mathbb{S}^{d-1}$ of radius θ, and

$$\triangle_\theta = \frac{\partial^2}{\partial\theta^2} + (d-2)\cot\theta\frac{\partial}{\partial\theta}.$$

See [12, p. 171–172]. Then

$$\triangle\varphi_j = \left(\frac{\partial^2}{\partial\theta^2} + (d-2)\cot\theta\frac{\partial}{\partial\theta}\right)\varphi(m\theta).$$

It follows that

$$\|(-\triangle)^v\varphi_j\|_\infty = \|\triangle^v\varphi_j\|_\infty = \|\triangle_\theta^v\varphi(m\theta)\|_\infty \leq c \cdot m^{2v}. \tag{5.11}$$

For a second continuously differentiable function f on $\mathbb{S}^{d-1}$, let $F(z) := f(z/|z|)$ be the function on $\mathbb{R}^n \setminus \{0\}$. Then for $x \in \mathbb{S}^{d-1}$,

$$\triangle f(x) = \sum_{j=1}^{d} \frac{\partial^2}{\partial z_j^2}F(z)\bigg|_{z=x}.$$

This means that supp $\triangle f \subset$ supp f, and hence supp $(-\triangle)^v\varphi_i \subset B\left(x_i, \frac{1}{m}\right)$, which together with (5.11) gives that

$$\|(-\triangle)^v f_a\|_p \ll m^{2v-\frac{d-1}{p}}\|a\|_{l_p^N} \ll m^{2v}\|f_a\|_p.$$

By a Kolmogorov type inequality (see [9, Theorem 8.1]), we get

$$\|f_a^{(l)}\|_p = \|(-\triangle)^{\frac{l}{2}} f_a\|_p \ll \|f_a\|_p^{1-\frac{l}{2\left[\frac{l}{2}\right]+2}} \|(-\triangle)^{1+\left[\frac{l}{2}\right]} f_a\|_p^{\frac{l}{2\left[\frac{l}{2}\right]+2}} \ll m^l \|f_a\|_p.$$

Since $\Omega(t) \in \Phi_l^*$, we know that there exists a $\beta,\ 0 < \beta < l$, for which $\Omega(t)/t^\beta$ is almost decreasing. This yields

$$I_2 \ll \sum_{s=J+1}^{+\infty} \frac{m^l \|f_a\|_p}{2^{sl}\Omega(2^{-s})} \ll \frac{\|f_a\|_p}{\Omega(2^{-J})} \sum_{s=J+1}^{+\infty} 2^{(J-s)(l-\beta)} \ll \frac{\|f_a\|_p}{\Omega(2^{-J})} \ll \frac{\|f_a\|_p}{\Omega\left(n^{-\frac{1}{d-1}}\right)}. \tag{5.12}$$

Obviously, (5.8) follows from (5.9), (5.10) and (5.12). Using (5.6)–(5.8), we get

$$S_n\left(BB^{\Omega}_{p,\theta}, L_q\right) \gg \Omega\left(n^{-\frac{1}{d-1}}\right) S_n(A_N \cap BL_p, L_q \cap A_N) \gg \Omega\left(n^{-\frac{1}{d-1}}\right) n^{\frac{1}{p}-\frac{1}{q}} S_n(B\ell_p^N, \ell_q^N).$$

Lemma 3 is now proved. $\square$

6. Proof of Theorem 1

Proof. The following proof is standard. First, by Lemma 3 and the following estimates (see [25, p. 236]):

$$d_n\left(B\ell_p^N, \ell_q^N\right) \gg \begin{cases} n^{\frac{1}{q}-\frac{1}{p}}, & 1 \le q \le p \le \infty; \\ n^{\frac{1}{q}-\frac{1}{p}}, & 2 \le p \le q \le \infty; \\ n^{\frac{1}{q}-\frac{1}{2}}, & 1 \le p \le 2 \le q \le \infty; \\ 1, & 1 \le p \le q \le 2. \end{cases} \tag{6.1}$$

we get the lower estimates for $d_n\left(BB^{\Omega}_{p,\theta}, L_q\right)$. Using Lemma 3, (6.1) and the following relations:

$$d^n(B\ell_p^N, \ell_q^N) = d_n(B\ell_{q'}^N, \ell_{p'}^N),$$
$$\delta_n\left(BB^{\Omega}_{p,\theta}, L_q\right) \ge \max\left\{d_n\left(BB^{\Omega}_{p,\theta}, L_q\right), d^n\left(BB^{\Omega}_{p,\theta}, L_q\right)\right\}, \tag{6.2}$$

we obtain the desired lower estimates for $d^n\left(BB^{\Omega}_{p,\theta}, L_q\right)$ and $\delta_n\left(BB^{\Omega}_{p,\theta}, L_q\right)$.

Next, we prove the upper estimates. For the Kolmogorov widths $d_n\left(BB^{\Omega}_{p,\theta}, L_q\right)$, in the region I: $1 \le q \le p \le \infty$ and the region II: $1 \le p \le q \le 2$, the upper estimates follow directly from Lemma 1. In the region IV: $1 \le p \le 2 \le q \le \infty$, we use Lemma 2. Define

$$n_j := \begin{cases} \#\Lambda_j^d, & \text{if } 0 \le j \le J; \\ \left[2^{J(d-1)(1+\rho)-(d-1)\rho j}\right], & \text{if } J < j \le J\left(1+\dfrac{1}{\rho}\right); \\ 0, & \text{if } j > J\left(1+\dfrac{1}{\rho}\right), \end{cases} \tag{6.3}$$

where $\rho > 0$ is sufficiently small. Since

$$\sum_{j=0}^{\infty} n_j \ll \sum_{0\le j\le J} 2^{j(d-1)} + \sum_{J<j\le J\left(1+\frac{1}{\rho}\right)} 2^{J(d-1)(1+\rho)-(d-1)\rho j} \ll 2^{J(d-1)},$$

there is a constant c independent of n such that $\sum_{j=0}^{\infty} n_j \le c2^{J(d-1)} \le n$, where $2^{J(d-1)} \asymp n$. If $j > J\left(1+\frac{1}{\rho}\right)$, then $d_{n_j}(B\ell_p^{\#\Lambda_j^d}, \ell_q^{\#\Lambda_j^d}) = 1$. Applying Lemma 2 and the following improved Kashin inequality (see [17, p. 465]):

$$d_m(B\ell_p^M, \ell_\infty^M) \le d_m(B\ell_2^M, \ell_\infty^M) \ll m^{-\frac{1}{2}} \left(\log_2\left(\frac{eM}{m}\right)\right), \quad 1 \le m \le M,$$

we get

$$\begin{aligned} d_n\left(BB_{p,\theta}^{\Omega}, L_q\right) &\ll d_n\left(BB_{p,\theta}^{\Omega}, L_\infty\right) \\ &\ll \sum_{J<j\le J\left(1+\frac{1}{\rho}\right)} \Omega(2^{-j})2^{j(d-1)/p} d_{n_j}(B\ell_p^{\#\Lambda_j^d}, \ell_\infty^{\#\Lambda_j^d}) + \sum_{j>J\left(1+\frac{1}{\rho}\right)} \Omega(2^{-j})2^{j(d-1)/p}. \\ &\ll \sum_{J<j\le J\left(1+\frac{1}{\rho}\right)} \Omega(2^{-j})2^{j(d-1)/p}2^{-[J(d-1)(1+\rho)-(d-1)j\rho]/2}(j-J)^{1/2} \\ &\quad + \sum_{j>J\left(1+\frac{1}{\rho}\right)} \Omega(2^{-j})2^{j(d-1)/p} := I_1 + I_2. \end{aligned} \tag{6.4}$$

If $\alpha > (d-1)/p$, using the fact that $\Omega(t)/t^\alpha$ is almost increasing, we obtain

$$\begin{aligned} I_1 &\ll \Omega(2^{-J}) \sum_{J<j\le\left(1+\frac{1}{\rho}\right)J} 2^{(J-j)\alpha}2^{j(d-1)/p}2^{-[J(d-1)(1+\rho)-(d-1)\rho j]/2}(j-J)^{\frac{1}{2}} \\ &= \Omega(2^{-J})2^{J(d-1)\left(\frac{1}{p}-\frac{1}{2}\right)} \sum_{J<j\le\left(1+\frac{1}{\rho}\right)J} 2^{-(j-J)(\alpha-\frac{d-1}{p}-\frac{\rho(d-1)}{2})}(j-J)^{\frac{1}{2}} \\ &\ll \Omega(2^{-J})2^{J(d-1)\left(\frac{1}{p}-\frac{1}{2}\right)} \ll \Omega\left(n^{-\frac{1}{d-1}}\right) n^{\frac{1}{p}-\frac{1}{2}}. \end{aligned} \tag{6.5}$$

Similarly, we can get

$$\begin{aligned} I_2 &\ll \Omega(2^{-J}) \sum_{j>J\left(1+\frac{1}{\rho}\right)} 2^{(J-j)\alpha+\frac{j(d-1)}{p}} \\ &\ll \Omega(2^{-J})2^{J\alpha}2^{-J\left(1+\frac{1}{\rho}\right)\left(\alpha-\frac{d-1}{p}\right)} \\ &\ll \Omega(2^{-J})2^{J(d-1)\left(\frac{1}{p}-\frac{1}{2}\right)} \ll \Omega\left(n^{-\frac{1}{d-1}}\right) n^{\frac{1}{p}-\frac{1}{2}}. \end{aligned} \tag{6.6}$$

Here we used the fact that $\rho > 0$ is sufficiently small. Then (6.6) combined with (6.4) and (6.5) gives the upper estimates for $d_n\left(BB_{p,\theta}^{\Omega}, L_q\right)$ in the region IV. In the region III: $2 \le p \le q \le \infty$, the upper estimates for the Kolmogorov widths follow from the fact that

$$d_n\left(BB_{p,\theta}^{\Omega}, L_q\right) \le d_n\left(BB_{2,\theta}^{\Omega}, L_\infty\right) \ll \Omega\left(n^{-\frac{1}{d-1}}\right).$$

For the linear widths $\delta_n\left(BB_{p,\theta}^{\Omega}, L_q\right)$, in the regions I, II, and III, the upper estimates follow directly from Lemma 1. In the region IV, we also use Lemma 2 with n_j given in (6.3). If $p = 1,\ q = p' = \infty,\ \alpha > d-1$, then the upper estimate for $\delta_n\left(BB_{p,\theta}^{\Omega}, L_q\right)$ follows from Lemma 2, (6.3)–(6.6), and the following equality (see [17, p. 412]):

$$\delta_m(B\ell_1^M, \ell_\infty^M) = d_m(B\ell_1^M, \ell_\infty^M), \quad 1 \le m \le M.$$

If $1 < p \le 2,\ q = p',\ \alpha > \left(\frac{1}{p} - \frac{1}{p'}\right)(d-1)$, then using Lemma 2, (6.3), $\Omega(t)/t^\alpha$ is almost increasing and the following Gluskin estimate (see [17, p. 473]):

$$\delta_m(B\ell_p^M, \ell_{p'}^M) \ll M^{1-\frac{1}{p}} m^{-\frac{1}{2}}, \quad 1 \le m \le M,$$

we get

$$\begin{aligned}\delta_n\left(BB^{\Omega}_{p,\theta},L_{p'}\right) &\ll \sum_{J<j\le J\left(1+\frac{1}{\rho}\right)} \Omega(2^{-j})2^{j(d-1)\left(\frac{1}{p}-\frac{1}{p'}\right)}2^{j(d-1)\left(1-\frac{1}{p}\right)}2^{-[J(d-1)(1+\rho)-(d-1)\rho j]/2}\\ &\quad+\sum_{j>J\left(1+\frac{1}{\rho}\right)} \Omega(2^{-j})2^{j(d-1)\left(\frac{1}{p}-\frac{1}{p'}\right)}\\ &\ll \Omega(2^{-J})2^{J(d-1)\left(\frac{1}{p}-\frac{1}{2}\right)} \ll \Omega\left(n^{-\frac{1}{d-1}}\right)n^{\frac{1}{p}-\frac{1}{2}}. \end{aligned} \tag{6.7}$$

Hence, in the region IV, the upper estimates for $\delta_n\left(BB^{\Omega}_{p,\theta},L_q\right)$ follow from (6.7) and the following relations:

$$\delta_n\left(BB^{\Omega}_{p,\theta},L_q\right) \ll \begin{cases}\delta_n\left(BB^{\Omega}_{p,\theta},L_{p'}\right), & \text{if } p'\ge q;\\ \delta_n\left(BB^{\Omega}_{q',\theta},L_q\right), & \text{if } p'\le q.\end{cases}$$

For the Gelfand widths $d^n\left(BB^{\Omega}_{p,\theta},L_q\right)$, in the regions I and III, the upper estimates follow directly from (5.2) and the proven upper estimates for the linear widths. In the region IV, if $\alpha > (1-1/q)(d-1)$, using Lemma 2, (6.3)–(6.6), and the following relation:

$$d^{n_j}(B\ell_1^{\#\Lambda_j^d},\ell_q^{\#\Lambda_j^d}) = d_{n_j}(B\ell_{q'}^{\#\Lambda_j^d},\ell_\infty^{\#\Lambda_j^d}),$$

we obtain

$$\begin{aligned} d^n\left(BB^{\Omega}_{p,\theta},L_q\right) &\le d^n\left(BB^{\Omega}_{1,\theta},L_q\right)\\ &\ll \sum_{J<j\le J\left(1+\frac{1}{\rho}\right)} \Omega(2^{-j})2^{j(d-1)(1-1/q)}d_{n_j}(B\ell_{q'}^{\#\Lambda_j^d},\ell_\infty^{\#\Lambda_j^d})\\ &\quad+\sum_{j>J\left(1+\frac{1}{\rho}\right)} \Omega(2^{-j})2^{j(d-1)(1-1/q)}\\ &\ll \Omega\left(n^{-\frac{1}{d-1}}\right)n^{\frac{1}{2}-\frac{1}{q}}. \end{aligned}$$

In the region II, it follows that

$$d^n\left(BB^{\Omega}_{p,\theta},L_q\right) \ll d^n\left(BB^{\Omega}_{1,\theta},L_2\right) \ll \Omega\left(n^{-\frac{1}{d-1}}\right).$$

Theorem 1 is now proved. $\square$

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